THE BEHAVIOR AT INFINITY OF CERTAIN CONVOLUTION TRANSFORMS(1)

BY

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1. **Introduction.** Let b, $\{a_k\}_1^\infty$ be real constants subject to the sole restriction that

(1) $$\sum_{k=1}^{\infty} a_k^{-2} < \infty,$$

and let

(2) $$E(s) = e^{bs}\prod_1^\infty\left(1 - \frac{s}{a_k}\right)e^{s/a_k}.$$

We define

(3) $$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{E(s)}ds \qquad (-\infty < t < \infty).$$

It may be verified that

(4) $$\int_{-\infty}^{\infty} G(t)e^{-st}ds = \frac{1}{E(s)},$$

the bilateral Laplace transform converging absolutely in the strip $(\alpha_1 < \sigma < \alpha_2)$ where

(5) $$\alpha_2 = \operatorname*{Min}_{a_k > 0} a_k, \qquad \alpha_1 = \operatorname*{Max}_{a_k < 0} a_k,$$

and $s = \sigma + i\tau$. The function $G(t)$ and the associated convolution transform

(6) $$f(x) = \int_{-\infty}^{\infty} G(x - t)d\alpha(t)$$

have been studied in [2], [3], and [4](2).

A kernel $G(t)$ belongs to class II if the constants a_k are positive and if $\sum_1^\infty 1/a_k = \infty$. If $G(t) \in$ class II and if the convolution transform (6) converges for any one value of x then it converges for all larger values of x. Consequently there exists a number γ_c, the abscissa of convergence, such that the

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transform (6) converges for $x>\gamma_c$ and diverges for $x<\gamma_c$.

Our principal result is that if $f(x)$ is the generating function of a convolution transform with class II kernel $G(t)$ and determining function $\alpha(t)$, and if

$$f(x) = O[G(x-\rho)] \qquad (x \to +\infty) \tag{7}$$

then $\alpha(t)$ is necessarily constant for $t>\rho$, so that

$$f(x) = \int_{-\infty}^{\rho+\epsilon} G(x-t)d\alpha(t)$$

for every $\epsilon>0$. If (7) holds for arbitrarily large negative ρ then $f(x)\equiv 0$.

As an example we set

$$E(s) = [\Gamma(1-s)]^{-1}, \qquad G(t) = e^{-e^t}e^t.$$

If

$$f(x) = \int_{-\infty}^{\infty} e^{-e^{x-t}}e^{x-t}\phi(t)dt \qquad (x > \gamma_c),$$

$$f(x) = O[e^{-e^{x-\rho}}e^{x-\rho}] \qquad (x \to +\infty),$$

then $\phi(t)=0$ for $(\rho<t<\infty)$. After an exponential change of variable this becomes a theorem concerning the Laplace transform. If

$$F(x) = \int_0^{\infty} e^{-xt}\Phi(t)dt \qquad (x > g_c),$$

$$F(x) = O(e^{-rx}) \qquad (x \to +\infty), \tag{8}$$

then $\Phi(t)=0$ for $(0<t<r)$. A proof of this special case has been given previously in [1].

A function $G(t)$ is said to belong to class Ia if there are both positive and negative a_k's and if $\sum_1^\infty 1/|a_k| = \infty$. If $G(t)\in$ class Ia and if the convolution transform (6) converges for any one value of x, it converges for all x. With each kernel $G(t)$ class Ia we may associate a class II kernel $\overline{G}(t)$ defined by the equations

$$\overline{E}(s) = e^{bs}\prod_1^\infty\left(1-\frac{s}{|a_k|}\right)e^{s/|a_k|},$$

$$\overline{G}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{\overline{E}(s)}ds \qquad (-\infty < t < \infty).$$

We shall also prove that if $f(x)$ is a generating function with kernel $G(t)\in$class Ia and if

$$f(x) = O[\overline{G}(x-\rho)] \qquad (x \to +\infty)$$

for arbitrarily large negative ρ, or

$$f(x) = O[\overline{G}(\rho - x)] \qquad (x \to -\infty)$$

for arbitrarily large positive ρ, then $f(x) \equiv 0$.

As an example of this result we may take

$$E(s) = \cos \pi s, \qquad G(t) = \operatorname{sech} \frac{t}{2} \cdot$$

After an exponential change of variables we obtain a theorem concerning the Stieltjes transform. If

$$F(x) = \int_{0+}^{\infty} \frac{\Phi(t)}{x+t}\, dt \qquad (0 < x < \infty),$$

and if

$$F(x) = O(e^{-rx^{1/2}}) \qquad (x \to +\infty)$$

or if

$$F(x) = O(e^{-rx^{-1/2}}) \qquad (x \to 0+)$$

for arbitrarily large r, then $F(x) \equiv 0$.

2. **A special case.** In this section we shall prove our first theorem under very special assumptions. The most general case can, however, be reduced to this one by elementary transformations.

THEOREM 2. *If*

1. $G(t) \in$ *class* II,
2. $\phi(t) \in L$ $(\rho \leqq t \leqq \infty)$,
3. $h(x) = \int_\rho^\infty G(x-t)e^{ct}\phi(t)dt$ $(c < \alpha_2)$, *where* α_2 *is defined as in* (5) §1,
4. $h(x) = O[G(x-\rho)]$ $\qquad (x \to +\infty)$,

then

$$\phi(t) \equiv 0 \qquad (\rho < t < \infty).$$

We define

$$\phi^*(s) = \int_\rho^\infty e^{ct}\phi(t)e^{-st}dt. \tag{1}$$

This bilateral Laplace transform converges absolutely for $(c \leqq \sigma < \infty)$. We know that

$$\frac{1}{E(s)} = \int_{-\infty}^{\infty} G(t)e^{-st}dt, \tag{2}$$

the integral converging absolutely for $(-\infty < \sigma/\alpha_2)$.

Since $G(t)e^{-ct}$ is positive and bell-shaped, see [3; §10], we have for x sufficiently large and negative

$$| h(x) | \leqq e^{cx}\left[\operatorname*{Max}_{\rho\leqq t\leqq\infty} G(x-t)e^{-c(x-t)}\right]\int_{\rho}^{\infty} | \phi(t) | \, dt$$
$$= O[G(x-\rho)] \qquad (x\to-\infty).$$

From this estimation and from assumption 4 it follows that

$$(3) \qquad | h(x) | \leqq O(1)G(x-\rho) \qquad (-\infty < x < \infty).$$

This implies that the bilateral Laplace transform

$$(4) \qquad h^*(s) = \int_{-\infty}^{\infty} h(x)e^{-sx}dx$$

converges absolutely for $(-\infty < \sigma < \alpha_2)$.

Since the integrals (1) and (2) have a common strip of absolute convergence $(c < \sigma < \alpha_2)$, the convolution theorem for the bilateral Laplace transform tells us that in this strip

$$h^*(s) = \frac{1}{E(s)}\phi^*(s),$$

or

$$(5) \qquad \phi^*(s) = h^*(s)E(s).$$

Since $E(s)$ is an entire function, equation (5) provides a continuation of $\phi^*(s)$ into the half-plane $\sigma \leqq c$, so that $\phi^*(s)$ is also an entire function.

By a change of variable in equation (1) we obtain

$$e^{\rho(s-c)}\phi^*(s) = \int_0^{\infty} e^{ct}\phi(t+\rho)e^{-st}dt.$$

It is clear that $e^{\rho s}\phi^*(s)$ is bounded in the half-plane $\sigma \geqq c$. In particular $e^{\rho s}\phi^*(s)$ is bounded on the line $(\sigma = c, \ -\infty < \tau < \infty)$.

Using inequality (3) we have

$$| h^*(s) | = O(1)[E(\sigma)e^{\rho\sigma}]^{-1} \qquad (-\infty < \sigma < \alpha_2)$$

from which we obtain

$$| e^{\rho s}\phi^*(s) | \leqq O(1)\left|\frac{E(s)}{E(\sigma)}\right| \qquad (-\infty < \sigma < \alpha_2).$$

It follows that $e^{\rho s}\phi^*(s)$ is bounded on the half-line $(-\infty < \sigma \leqq c, \tau = 0)$ and that in the half-plane $(-\infty < c \leqq \sigma, \ \phi^*(s))$ is at most of order two minimal type.

We now know that $e^{\rho s}\phi^*(s)$ is bounded on the lines $(\sigma = c, \ -\infty < \tau < \infty)$ and $(-\infty < \sigma \leqq c, \ \tau = 0)$ and is at most of order two minimal type for $\sigma \leqq c$.

Applying a familiar form of the Phragmén-Lindelöf principle to each of the two quadrants of this configuration we find that $e^{\rho s}\phi^*(s)$ is bounded for $\sigma \leqq c$ and thus in the entire plane. By Liouville's theorem $e^{\rho s}\phi^*(s)$, being entire and bounded, is a constant. Since

$$\begin{aligned} e^{i\rho\tau}\phi^*(c+i\tau) &= \int_0^\infty \phi(t+\rho)e^{-it\tau}dt \\ &= o(1) \qquad\qquad (\tau \to \pm\infty) \end{aligned}$$

it follows that $\phi^*(s) \equiv 0$. This implies that $\phi(t) = 0$ almost everywhere $(\rho \leqq t < \infty)$. q.e.d.

3. **Two lemmas.** We consider here two lemmas necessary to reduce the general case to the special case which we have just considered. Let

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t) \tag{1}$$

have abscissa of convergence γ_c. We set

$$\begin{aligned} A_1(t) &= a_1 e^{a_1 t}\int_t^\infty e^{-a_1 u}d\alpha(u), \\ A_m(t) &= a_m e^{a_m t}\int_t^\infty e^{-a_m u}A_{m-1}(u)du \qquad (m = 2, \cdots, n) \end{aligned} \tag{2}$$

where n is chosen so large that $s-\alpha_2$ is not a zero of

$$\prod_{n+1}^{\infty}\left(1-\frac{s}{a_k}\right)e^{s/a_k}.$$

We further define the auxiliary kernel

$$H(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp\left(s\left(t-\sum_1^n a_k^{-1}\right)\right)}{e^{bs}\prod_{n+1}^{\infty}(1-(s/a_k))e^{s/a_k}}ds \qquad (-\infty < t < \infty). \tag{3}$$

Lemma 3a. *If $f(x)$ is given by* (1) *and if $A_n(t)$ and $H(t)$ are defined as in* (2) *and* (3) *then*

A. $$f(x) = \int_{-\infty}^{\infty} H(x-t)A_n(t)dt \qquad (x > \gamma_c),$$

B. $$A_n(t) = o(e^{\alpha_2 t}) \qquad (t \to +\infty).$$

The convergence of the integral (1) insures that $A_1(t)$ is defined, see [3; §15]. Writing

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t) = \int_{-\infty}^{\infty} G(x-t)e^{a_1 t}e^{-a_1 t}d\alpha(t)$$

and integrating by parts we have

$$\int_{-\infty}^{\infty} G(x-t)d\alpha(t) = \left[-G(x-t)e^{a_1 t}\int_t^{\infty} e^{-a_1 u}d\alpha(u)\right]_{-\infty}^{\infty}$$
$$+ \int_{-\infty}^{\infty} A_1(t)\left[\frac{1}{a_1}e^{-a_1 t}\frac{d}{dt}e^{a_1 t}G(x-t)\right]dt.$$

It is easily verified that for $x>\gamma_c$ the integrated term vanishes; see [3; §§16, 20]. Thus

$$f(x) = \int_{-\infty}^{\infty}\left\{\left[1+\frac{D}{a_1}\right]G(x-t)\right\}A_1(t)dt \qquad (x>\gamma_c). \tag{4}$$

The convergence of the integral (4) insures that $A_2(t)$ is defined, and so forth. After n such steps we obtain

$$f(x) = \int_{-\infty}^{\infty}\left\{\prod_1^n\left(1+\frac{D}{a_k}\right)G(x-t)\right\}A_n(t)dt$$
$$= \int_{-\infty}^{\infty} H(x-t)A_n(t)dt,$$

which is conclusion A.

We may establish conclusion B by induction. We know that $I(t) = \int_t^{\infty} e^{-\alpha_2 u}d\alpha(u)$ converges. If $\alpha_2 = a_1$, conclusion B is obvious for $n=1$. If $\alpha_2 < a_1$, then

$$A_1(t) = a_1 e^{a_1 t}\left\{\left[-e^{-a_1 u+\alpha_2 u}I(u)\right]_t^{\infty} + (\alpha_2 - a_1)\int_t^{\infty} e^{-a_1 u+\alpha_2 u}I(u)du\right\},$$
$$= o(e^{\alpha_2 t}) \qquad (t\to+\infty).$$

Suppose now that our result is proved for $m-1$. We will establish it for m. If $a_m=\alpha_2$ it is obvious. If $a_m>\alpha_2$, then using our induction assumption we see that

$$A_m = o\left(e^{a_m u}\int_t^{\infty} e^{-a_m u}e^{\alpha_2 u}du\right)$$
$$= o(e^{\alpha_2 t}) \qquad (t\to+\infty).$$

Conclusion B now follows.

LEMMA 3b. *If*

1. $G(t)\in$ *class* II,
2. $f(x)=\int_{-\infty}^{\infty}G(x-t)d\alpha(t)$ *has abscissa of convergence* γ_c,

then

$$\int_{-\infty}^{\rho} G(x-t)d\alpha(t) = O[G(x-\rho)] \qquad (x\to+\infty).$$

If δ is any number greater than γ_c then we know, see [3; §§16, 20], that there is a positive constant A such that

$$|\alpha(t)| \leqq Ae^{\chi(\delta-t)} \qquad (-\infty < t \leqq \rho),$$

where $\chi(t) = -\log G(t)$. Integrating by parts we find that for $x > \delta$

$$\int_{-\infty}^{\rho} G(x-t)d\alpha(t) = \int_{-\infty}^{\rho} \chi'(x-t)e^{-\chi(x-t)}\alpha(t)dt + O[G(x-\rho)].$$

For x sufficiently large $\chi'(x-t)$ is of constant (positive) sign in $(-\infty < t < \rho)$ therefore

$$\left|\int_{-\infty}^{\rho} \chi'(x-t)e^{-\chi(x-t)}\alpha(t)dt\right| \leqq A\int_{-\infty}^{\rho} \chi'(x-t)e^{-\chi(x-t)+\chi(\delta-t)}dt.$$

Integrating by parts again,

$$\int_{-\infty}^{\rho} \chi'(x-t)e^{-\chi(x-t)+\chi(\delta-t)}$$
$$= \left[e^{-\chi(x-t)+\chi(\delta-t)}\right]_{-\infty}^{\rho} + \int_{-\infty}^{\rho} \chi'(\delta-t)e^{-\chi(x-t)+\chi(\delta-t)}dt.$$

Now

$$\left[e^{-\chi(x-t)+\chi(\delta-t)}\right]_{-\infty}^{\rho} = G(x-\rho)/G(\delta-\rho).$$

Further if $(\theta > \delta > \gamma_c)$, then

$$\int_{-\infty}^{\rho} \chi'(\delta-t)e^{-\chi(x-t)+\chi(\delta-t)}dt$$
$$= \int_{-\infty}^{\rho} \chi'(\delta-t)e^{\chi(\delta-t)-\chi(\theta-t)}e^{\chi(\theta-t)-\chi(x-t)}dt$$
$$\leqq \sup_{-\infty<t\leqq\rho} [G(x-t)/G(\theta-t)] \int_{-\infty}^{\rho} \left|\chi'(\delta-t)\right| e^{\chi(\delta-t)-\chi(\theta-t)}dt.$$

It may be verified that for x sufficiently large,

$$\sup_{-\infty<t\leqq\rho} [G(x-t)/G(\theta-t)] = G(x-\rho)/G(\theta-\rho),$$

see [3; §20]. Thus

$$\int_{-\infty}^{\rho} \left| \chi'(\delta - t) \right| e^{\chi(\delta - t) - \chi(x - t)} dt = O[G(x - \rho)] \qquad (x \to +\infty).$$

Combining our results we have as desired,

$$\int_{-\infty}^{\rho} G(x - t) d\alpha(t) = O[G(x - \rho)] \qquad (x \to +\infty).$$

4. **The general case, $G(t) \in$ class II.**

THEOREM 4. *If*
1. $G(t) \in$ *class* II,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$ *has abscissa of convergence* γ_c,
3. $f(x) = O[G(x-\rho)]$ $(x \to +\infty)$,

then $\alpha(t)$ *is constant for* $\rho < t < \infty$.

By conclusion A of Lemma 3a we have

$$f(x) = \int_{-\infty}^{\infty} H(x - t) A_n(t) dt$$

where H and A_n are defined in §3. It is known [3; §§20, 21] that if $\epsilon > 0$

$$\lim_{x \to +\infty} G(x - \rho)/H(x - \rho - \epsilon) = 0.$$

Thus

(1) $$f(x) = O[H(x - \rho - \epsilon)] \qquad (x \to +\infty).$$

By Lemma 3b and equation (1)

$$\int_{\rho+\epsilon}^{\infty} H(x - t) A_n(t) dt = O[H(x - \rho - \epsilon)] \qquad (x \to +\infty).$$

If we choose c $(\alpha_2 < c < \mathrm{Min}_{k>n}\, a_k)$, then by conclusion B of Lemma 3a

$$\int_{\rho+\epsilon}^{\infty} H(x - t) A_n(t) dt = \int_{\rho+\epsilon}^{\infty} H(x - t) e^{ct}[e^{-ct} A_n(t)] dt,$$

where $e^{-ct} A_n(t) \in L(\rho + \epsilon, \infty)$. We may apply Theorem 2 to conclude that $A_n(t) = 0$ $(\rho + \epsilon < t < \infty)$. Since ϵ is arbitrary $A_n(t) = 0$ $(\rho < t < \infty)$. Using the definition of $A_n(t)$ it is easily seen that $\alpha(t)$ is constant for $(\rho < t < \infty)$. q.e.d.

5. $G(t) \in$ **class** Ia. In order to illustrate the essential point of the argument which follows let us consider a very special case. Let $G(t) \in$ class Ia and let

(1) $$f(x) = \int_{-\infty}^{\infty} G(x - t) \phi(t) dt$$

where $\phi(t) \in L^2(-\infty, \infty)$. We shall show that if

$$f(x) = O(\overline{G}(x - \rho)) \qquad (x \to +\infty) \tag{2}$$

for arbitrarily large negative ρ then $f(x) \equiv 0$. Let $\mathfrak{F}$ denote the Fourier transformation

$$\mathfrak{F}_t\psi = \frac{1}{(2\pi)^{1/2}} \operatorname*{l.i.m.}_{T\to\infty}^{(2)} \int_{-T}^{T} \psi(u)e^{itu}du,$$

and let $\mathfrak{F}_t^{-1}$ denote its inverse

$$\mathfrak{F}_t^{-1}\psi = \frac{1}{(2\pi)^{1/2}} \operatorname*{l.i.m.}_{T\to\infty}^{(2)} \int_{-T}^{T} \psi(u)e^{-itu}du.$$

We assert that

$$f(x) = \int_{-\infty}^{\infty} \overline{G}(x - t)\left[\mathfrak{F}_t^{-1}\left(\frac{E(-iu)}{\overline{E}(-iu)}\,\mathfrak{F}_u\phi(t)\right)\right]dt. \tag{3}$$

From equation (1) we see that the Fourier transform of $f(x)$ satisfies the equation $\mathfrak{F}_u f = E(-iu)\mathfrak{F}_u\phi$. Computing the Fourier transform of the right-hand side of equation (3) we obtain

$$\overline{E}(-iu)\mathfrak{F}_u\mathfrak{F}_t^{-1}\left[\frac{E(-iu)}{\overline{E}(-iu)}\,\mathfrak{F}_u\phi\right] = E(-iu)\mathfrak{F}_u\phi = \mathfrak{F}_u f.$$

The validity of equation (3) follows. By equations (2) and (3) and Theorem 4 $f(x) \equiv 0$, which is what we wished to show. The transformation

$$\phi(t) \to \mathfrak{F}_t^{-1}\left[\frac{E(-iu)}{\overline{E}(-iu)}\,\mathfrak{F}_u\phi\right]$$

is a Watson transformation. Our procedure in the general case is an adaptation of this argument.

LEMMA 5a. *If*

1. $G(t) \in$ *class* Ia,
2. $|\beta_1|, |\beta_2| < \text{Min}\,[-\alpha_1, \alpha_2]$,
3. $\phi(t) = O(e^{\beta_2 t})\ (t \to +\infty) = O(e^{\beta_1 t})\ (t \to -\infty)$,
4. $f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)dt$,
5. $f(x) = O[\overline{G}(x - \rho)]$ *as* $x \to +\infty$ *for arbitrarily large negative* ρ,

then $f(x) \equiv 0$.

Choose any constant c, $c \geqq 2\alpha_2$. We define

$$\begin{aligned} F(s) = \exp\left\{[b + (|a_1| + c)^{-1} + (|a_2| + c)^{-1}]s\right\} \\ \cdot \prod_{n=3}^{\infty}\left(1 - \frac{s}{|a_n| + c}\right)e^{s/(|a_n|+c)}, \end{aligned} \tag{1}$$

$$(2) \qquad K(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)}\, ds \qquad (-\infty < t < \infty),$$

$$(3) \qquad J(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{F(s)}{E(s)}\, e^{st} ds \qquad (-\infty < t < \infty).$$

The function $K(t)$ is a class II kernel. Let $\eta > 0$ be so small that $\alpha_1 + \eta < \beta_1$ and $\alpha_2 - \eta > \beta_2$. By [3; §9]

$$(4) \qquad \begin{aligned} K(t) &= O\left[e^{(\alpha_2-\eta)t}\right] && (t \to -\infty) \\ &= O\left[e^{-kt}\right] && (t \to +\infty) \end{aligned}$$

for every positive K. Further,

$$(5) \qquad \int_{-\infty}^{\infty} e^{-st} K(t) dt = \frac{1}{F(s)},$$

the bilateral Laplace transform converging absolutely for $(-\infty < \sigma < \alpha_2)$.

For c chosen as above and for any σ $(\alpha_1 < \sigma < \alpha_2)$, it may be verified that

$$\left| 1 - \frac{\sigma + i\tau}{|a_n| + c} \right| \left| 1 - \frac{\sigma + i\tau}{a_n} \right|^{-1}$$

decreases as $|\tau|$ increases. Let $\alpha_1 < \sigma_1 \leqq \sigma \leqq \sigma_2 < \alpha_2$ be any proper subinterval of $(\alpha_1 < \sigma < \alpha_2)$, and let

$$M = \operatorname*{Max}_{\sigma_1 \leqq \sigma \leqq \sigma_2} |F(\sigma)/E(\sigma)|.$$

We have

$$\left| \frac{F(s)}{E(s)} \right| \leqq \left| \left(1 - \frac{s}{a_1}\right)\left(1 - \frac{s}{a_2}\right) \right|^{-1} M \qquad (\sigma_1 \leqq \sigma \leqq \sigma_2).$$

It follows that

$$(6) \qquad \left| \frac{F(s)}{E(s)} \right| = O\left(\frac{1}{\tau^2}\right) \qquad (\tau \to \pm \infty)$$

uniformly in σ for σ in any proper subinterval of $(\alpha_1 < \sigma < \alpha_2)$. Because of this order relation we may deform the line of integration of the integral (3) to $\sigma = \alpha_2 - \eta$ or to $\sigma = \alpha_1 + \eta$. We obtain

$$\begin{aligned} J(t) &= \frac{1}{2\pi i}\int_{\alpha_2-\eta-i\infty}^{\alpha_2-\eta+i\infty} \frac{F(s)}{E(s)}\, e^{st} ds \\ &= \frac{1}{2\pi i}\int_{\alpha_1+\eta-i\infty}^{\alpha_1+\eta+i\infty} \frac{F(s)}{E(s)}\, e^{st} ds, \end{aligned}$$

and these together with equation (6) imply that

$$\begin{aligned} J(t) &= O(e^{(\alpha_2-\eta)t}) & (t \to -\infty) \\ &= O(e^{(\alpha_1+\eta)t}) & (t \to +\infty). \end{aligned} \tag{7}$$

Further, a simple application of a theorem of Hamburger, see [6; pp. 265–266]shows that

$$\int_{-\infty}^{\infty} J(t)e^{-st}dt = \frac{F(s)}{E(s)}, \tag{8}$$

the integral converging absolutely for $(\alpha_1 < \sigma < \alpha_2)$.

From equations (5) and (8) and the convolution theorem for the bilateral Laplace transform we have

$$\int_{-\infty}^{\infty} K(x-t)J(t)dt = G(x) \qquad (-\infty < x < \infty). \tag{9}$$

Consequently we may write

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt = \int_{-\infty}^{\infty} \phi(t)dt \int_{-\infty}^{\infty} K(x-u)J(u-t)du. \tag{10}$$

Because of the order conditions (4) and (7) and assumption 3 we may invert the order of integration to obtain

$$f(x) = \int_{-\infty}^{\infty} K(x-u)\Phi(u)du, \tag{11}$$

where

$$\Phi(u) = \int_{-\infty}^{\infty} J(u-t)\phi(t)dt.$$

Consider the class II kernel $L(t)$,

$$L(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{e^{bs}\prod_1^\infty (1 - s/(|a_k| + c))e^{s/(|a_k|+c)}} ds \qquad (-\infty < t < \infty).$$

It may be shown, see [3; §20], that

$$\bar{G}(t) = \left[\prod_1^\infty \left(1 + \frac{c}{|a_k|}\right)e^{-c/|a_k|}\right]^{-1} e^{-ct}L\left(t - \sum_1^\infty \frac{c}{|a_k|(|a_k| + c)}\right).$$

Assumption 5 thus implies that $f(x) = O[L(x-\rho)]$ $(x \to +\infty)$ for arbitrarily large negative ρ. It follows from [3; §§20, 21] that if $\epsilon > 0$ then

$$\lim_{t\to\infty} K(t)/L(t+\epsilon) = \infty.$$

Hence

$$f(x) = O[K(x-\rho)] \qquad (x \to +\infty) \tag{12}$$

for arbitrarily large negative ρ. Equations (11) and (12) and Theorem 4 imply that

$$f(x) \equiv 0,$$

which is what we wished to prove.

THEOREM 5b. *If*
1. $G(t) \in$ *class* Ia,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$,
3. $f(x) = O[\overline{G}(x-\rho)]$ $(x \to +\infty)$ *for arbitrarily large negative* ρ *where* $\overline{G}$ *is defined as in* §1,

then

$$f(x) \equiv 0.$$

Choose n so large that $s-\alpha_1$ and $s-\alpha_2$ are no longer zeros of

$$\prod_{n+1}^{\infty}\left(1 - \frac{s}{a_k}\right)e^{s/a_k},$$

and so large that if $\alpha_3 = \text{Min } |a_k|$, $k = n+1, n+2, \cdots$, then $\alpha_3 > \text{Max } [-\alpha_1, \alpha_2]$. Let

$$H(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp\left(s\left[t - \sum_1^n a_k^{-1}\right]\right)}{e^{bs}\prod_{n+1}^{\infty}(1 - s/a_k)e^{s/a_k}}\, ds,$$

$$\overline{H}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp\left(s\left[t - \sum_1^n a_k^{-1}\right]\right)}{e^{bs}\prod_{n+1}^{\infty}(1 - s/|a_k|)e^{s/|a_k|}}\, ds.$$

Further we set

$$A_1(t) = a_1 e^{a_1 t}\int_t^{\infty} e^{-a_1 u}d\alpha(u) \qquad (\text{if } a_1 > 0)$$

$$= a_1 e^{a_1 t}\int_{-\infty}^{t} e^{-a_1 u}d\alpha(u) \qquad (\text{if } a_1 < 0),$$

$$A_m(t) = a_m e^{a_m t}\int_t^{\infty} e^{-a_m u}A_{m-1}(u)du \qquad (\text{if } a_m > 0)$$

$$= a_m e^{a_m t}\int_{-\infty}^{t} e^{-a_m u}A_{m-1}(u)du \qquad (\text{if } a_m < 0),$$

for $m = 2, \cdots, n$. Just as in Lemma 3 we may show that

$$f(x) = \int_{-\infty}^{\infty} H(x-t)A_n(t)dt$$

and that

$$A_n(t) = o(e^{\alpha_2 t}) \qquad (t \to +\infty)$$
$$= o(e^{\alpha_1 t}) \qquad (t \to -\infty).$$

By our choice of n

$$H(t) = O(e^{-\alpha_3 |t|}) \qquad (t \to +\infty)$$

where $\alpha_3 > \text{Max} [-\alpha_1, \alpha_2]$. It may be shown, see [3; §§20, 21] that if $\epsilon > 0$ then

$$\lim_{x \to +\infty} \overline{G}(t)/\overline{H}(t-\epsilon) = 0.$$

Hence assumption 3 implies that

$$f(x) = O(\overline{H}(x-\rho)) \qquad (x \to +\infty)$$

for arbitrarily large negative ρ. We may now apply Lemma 5a to conclude that $f(x) \equiv 0$.

Similarly we may prove

THEOREM 5c. *If*

1. $G(t) \in$ *class* Ia,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$,
3. $f(x) = \overline{G}(\rho - x)$ $(x \to -\infty)$ *for arbitrarily large positive* ρ,

then

$$f(x) \equiv 0.$$

6. **Class III kernels.** A kernel $G(t)$ is said to belong to class III if the corresponding product $E(s)$ has only positive zeros and if $\sum_1^\infty a_k^{-1} < \infty$. The following result may be proved exactly as Theorem 4 was proved. The analogue of Theorem 2 which is needed here reduces to a special case of a well known result, see [5; pp. 322–327].

THEOREM 6. *If*

1. $G(t) \in$ *class* III,
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$ *is defined for* $x > T + b + \sum_1^\infty a_n^{-1}$,
3. $f(x) = 0$ $(x > \rho + b + \sum_1^\infty a_n^{-1}, \rho \geqq T)$,

then $\alpha(t)$ *is constant for* $(\rho < t < \infty)$.

BIBLIOGRAPHY

1. I. I. Hirschman, Jr., *A new representation and inversion theory for the Laplace integral*, Duke Math. J. vol. 15 (1948) pp. 473–494.

2. I. I. Hirschman, Jr., and D. V. Widder, *An inversion and representation theory for convolu-*

tion transforms with totally positive kernels, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 152–156.

3. ———, *The inversion of a general class of convolution transforms*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201.

4. I. J. Schoenberg, *On totally positive functions, Laplace integrals, and entire functions of the Laguerre-Pólya-Schur type*, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 11–17.

5. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford University Press, 1937.

6. D. V. Widder, *The Laplace transform*, Princeton University Press, 1941.

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